

THE TOP CHERN CLASS OF THE HODGE BUNDLE ON THE MODULI SPACE OF ABELIAN VARIETIES

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ABSTRACT. We give upper and lower bounds for the order of the top Chern class of the Hodge bundle on the moduli space of principally polarized abelian varieties. We also give a generalization to higher genera of the famous formula $12\lambda_1 = \delta$ for genus 1.

1. INTRODUCTION

Let \mathcal{A}_g/\mathbb{Z} denote the moduli stack of principally polarized abelian varieties of dimension g . This is an irreducible algebraic stack of relative dimension $g(g+1)/2$ with irreducible fibres over \mathbb{Z} . The stack \mathcal{A}_g carries a locally free sheaf \mathbb{E} of rank g , the Hodge bundle, defined as follows. If A/S is a principally polarized abelian variety with zero section s we get a locally free sheaf $s^*\Omega_{A/S}^1$ of rank g on S and this is compatible with pull backs. If $\pi : A \rightarrow S$ denotes the structure map it satisfies the property $\Omega_{A/S}^1 = \pi^*(\mathbb{E})$. The Hodge bundle can be extended to a locally free sheaf (again denoted by) \mathbb{E} on every smooth toroidal compactification $\tilde{\mathcal{A}}_g$ of \mathcal{A}_g , cf. [2].

The Chern classes λ_i of the Hodge bundle \mathbb{E} are defined over \mathbb{Z} and give for each fibre $\mathcal{A}_g \otimes k$ rise to classes λ_i in the Chow ring $CH^*(\mathcal{A}_g \otimes k)$, and in $CH^*(\tilde{\mathcal{A}}_g \otimes k)$. They generate subrings (\mathbb{Q} -subalgebras) of $CH_{\mathbb{Q}}^*(\mathcal{A}_g \otimes k)$ and of $CH_{\mathbb{Q}}^*(\tilde{\mathcal{A}}_g \otimes k)$ which are called the *tautological subrings*.

It was proved in [3] by an application of the Grothendieck-Riemann-Roch theorem that these classes in the Chow ring $CH_{\mathbb{Q}}^*(\mathcal{A}_g)$ with rational coefficients satisfy the following relation

$$(1.1) \quad (1 + \lambda_1 + \dots + \lambda_g)(1 - \lambda_1 + \dots + (-1)^g \lambda_g) = 1.$$

Furthermore, it was proved that λ_g vanishes in the Chow group $CH_{\mathbb{Q}}(\mathcal{A}_g)$ with rational coefficients. The class λ_g does not vanish on $\tilde{\mathcal{A}}_g$. This raises two questions. First, since λ_g is a torsion class on \mathcal{A}_g we may ask what its order is. Second, since λ_g up to torsion comes from a class on the ‘boundary’ $\tilde{\mathcal{A}}_g - \mathcal{A}_g$ we may ask for a description of this class. As an answer to these questions we give an upper bound on the order of λ_g in the second section and a non-vanishing result in the third section which implies a lower bound. We then give a several representing cycles for the top Chern class: on \mathcal{A}_g , on the Satake compactification, and in the last section we describe a cycle class in a partial compactification of \mathcal{A}_g which represents λ_g in the Chow group $CH_{\mathbb{Q}}^g$. This result can be viewed as a generalization of the well-known relation $12\lambda_1 = \delta$ for $g = 1$.

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2. A BOUND ON THE ORDER OF THE CLASS λ_g .

We will make some computations in the Chow group of \mathcal{A}_g . For the definition of the (integral) Chow groups for stacks we refer to [7]. We begin with a lemma which is no doubt well known.

Lemma 2.1. *If E is a vector bundle of rank g , the total Chern class of the graded vector bundle Λ^*E is zero in degrees 1 to $g-1$ and $-(g-1)!c_g$ in degree g .*

Proof. As usual we note that the components of the total Chern class are universal polynomials in the Chern classes $c_i := c_i(E)$ and we may let E be the universal bundle. Then as we may think of the coefficients in the universal polynomials as rational numbers we can note that the Chern classes of degree 1 to $g-1$ of Λ^*E vanish if and only if the Newton polynomials of the same degrees do, that is, if and only if $ch(\Lambda^*E)$ vanishes in the same degrees. Thus the first part followed from the Borel-Serre formula (cf. [1])

$$ch(\Lambda^*E) = (-1)^g c_g \text{Td}(E)^{-1}.$$

Furthermore, if the Chern classes of degree 1 to $g-1$ vanish for a bundle F , then it is clear that $(-1)^{g-1}g c_g(F) = s_g(F)$, as can be seen from Newton's formula. Hence in degree g we have $ch(\Lambda^*E)$ is $(-1)^{g-1}g c_g(F)/g!$ and using the Borel-Serre formula again gives the desired formula. \square

Lemma 2.2. *Let p be a prime. Suppose that $\pi : A \rightarrow S$ is a family of abelian varieties of relative dimension g , where S is a scheme, and L is a line bundle on A of order p , on all fibres of π . If $p > \min(2g, \dim S + g)$ then $p(g-1)!\lambda_g = 0$.*

Proof. By twisting L by a line bundle on S so that it is trivial along the zero section we may assume that it is of order p on A . Denoting the class of L in K_0 by $[L]$ we then either have that $p([L] - 1)$ has support of codimension $> 2g$ if $p > 2g$ or is zero if $p > \dim S + g$. Indeed, from the relation

$$0 = [L]^p - 1 = p([L] - 1)(1 + (p-1)/2([L] - 1) + \dots) + ([L] - 1)^p$$

and the fact that $[L] - 1$ is nilpotent we get that $p([L] - 1)$ is divisible by $([L] - 1)^p$ and that element is supported in codimension $\geq p$. Now in the first case the image under π of the support has codimension $> g$ on S and so we may safely remove it and may assume that $p[L] = p$ in $K_0(A)$.

Consider now the Poincaré bundle \mathcal{P} on $A \times_S \check{A}$, where \check{A} denotes the dual abelian variety. By base change $R\pi_*\mathcal{O}_A$ is the (derived) pullback along the zero-section of \check{A} of the sheaf $R\pi_*\mathcal{P}$. We have that $p[\mathcal{P}] = p[L \otimes \mathcal{P}]$ and so $p[R\pi_*\mathcal{P}] = p[R\pi_*(L \otimes \mathcal{P})]$. Now, a fibrewise calculation shows that $R\pi_*(L \otimes \mathcal{P})$ has support along the inverse of the section of \check{A} corresponding to L . As that section is everywhere disjoint from the zero section the pullback of $R\pi_*(L \otimes \mathcal{P})$ along the zero section is 0 and thus $p[R\pi_*\mathcal{O}_A] = 0$. Using lemma 2.1 and applying the total Chern class then gives $1 = (1 - (g-1)!\lambda_g + \dots)^p$ which in turn gives the lemma. \square

Definition-Lemma 2.3. *For an integer g we let n_g be the largest common divisor of all $p^{2g} - 1$ where p runs through all primes larger than a sufficiently large fixed number N (which may be taken to be $2g+1$). For an odd prime p , the exact power p^k of p that divides n_g is the largest k such that $p^{k-1}(p-1)$ divides $2g$ and 0 if $p-1$ does not divide $2g$. The exact power 2^k that divides n_g is the largest k such that 2^{k-2} divides $2g$.*

Proof. This follows directly from the structure of $(\mathbb{Z}/p^k)^*$ and Dirichlet's prime number theorem. \square

Example 2.4. We have $n_1 = 24$, $n_2 = 240$, $n_3 = 504$ and $n_4 = 480$.

Lemma 2.5. *We have*

$$\prod_{i=1}^g n_i = \prod_p ([\frac{2gp}{p-1}]!)_p,$$

where p runs over the primes.

Define for $g \geq 1$ the positive rational number

$$p(g) := (-1)^g \prod_{j=1}^g \frac{\zeta(1-2j)}{2},$$

where $\zeta(s)$ is the Riemann zeta-function. By the Proportionality theorem of Hirzebruch-Mumford and a theorem of Siegel-Harder we know that the degree of $\prod_{j=1}^g \lambda_j$ equals $\#\mathrm{Sp}(2g, \mathbb{Z}/n) \times p(g)$ on the scheme $\mathcal{A}_g[n]$ of abelian varieties with level n structure. For $n \geq 3$ this degree must be an integer. This implies the following corollary.

Corollary 2.6. *The rational number $\prod_{i=1}^g n_i$ is a multiple of the denominator of $p(g)$.*

Proposition 2.7. *Suppose that $\pi : A \rightarrow S$ is a family of abelian varieties of relative dimension g , where S is a scheme. Then $(g-1)!n_g\lambda_g = 0$ on S .*

Proof. For any prime p larger than $2g$ we can apply lemma 2.2 on the cover obtained by adding a line bundle everywhere of order p . Projecting down to S again and using that that cover has degree $p^{2g} - 1$ gives $(g-1)!p(p^{2g} - 1)\lambda_g = 0$. We then finish by using Definition 2.3 (and noting that the factor p causes no trouble as by using several primes we see that no prime $> 2g$ can divide the smallest annihilating integer). \square

Proposition 2.8. *Suppose that $\pi : A \rightarrow S$ is the universal family of abelian varieties of relative dimension g . Then $(g-1)!\prod_{i=1}^g n_i\lambda_g = 0$ on S .*

Proof. The proof is almost the same as that of 2.7 only that now we consider instead the cover given by putting a full level p -structure on A . This time we therefore get that $(g-1)!p^m(p^{2g} - 1)(p^{2(g-1)} - 1) \dots \lambda_g$, where m is some irrelevant positive integer. Using again 2.3 we conclude. \square

Example 2.9. i) For $g = 1$ we get $24\lambda_1 = 0$ which is off by a factor 2.

ii) For $g = 2$ we get $24 \cdot (16 \cdot 3 \cdot 5)\lambda_2 = 0$.

iii) For $g = 3$ we get $2 \cdot 24 \cdot (16 \cdot 3 \cdot 5) \cdot (8 \cdot 9 \cdot 7)\lambda_3 = 0$.

3. THE ORDER OF THE CHERN CLASSES OF THE DE RHAM BUNDLE

We will now consider the Chern classes of the bundle of first relative de Rham cohomology of the universal abelian variety over \mathcal{A}_g . We will apply our results to obtain information on the order of λ_g but the results we obtain should be of independent interest. Note that the *de Rham bundle*, $H_{dR}^1 := R^1\pi_*\Omega_{\mathcal{X}_g/\mathcal{A}_g}$, where $\pi: \mathcal{X}_g \rightarrow \mathcal{A}_g$ is the universal family, is provided with an integrable connection and hence its Chern classes in integral or ℓ -adic cohomology¹, which we will denote r_i , are torsion classes. In this section we will determine their exact order.

We begin by using a result of Grothendieck to get an upper bound for the order of r_i .

Proposition 3.1. *i) We have $r_i = 0$ for odd i .
ii) We have $n_i r_{2i} = 0$.*

Proof. The first part follows immediately because H_{dR}^1 is a symplectic vector bundle.

As for the second part we may assume that the characteristic is 0 as the case of positive characteristic follows from the characteristic 0 case by specialisation. In that case we may further reduce to the case of the base field being the complex number. We may also prove the annihilation of $n_i r_{2i}$ in ℓ -adic cohomology for a specific (but arbitrary) prime ℓ .

Now, the existence of Gauss-Manin connection on H_{dR}^1 means that it has a discrete structure group. More precisely, the fundamental group of the algebraic stack \mathcal{A}_g is $\mathrm{Sp}_{2g}(\mathbb{Z})$ and H_{dR}^1 is the vector bundle associated to the representation of it given by the natural inclusion of $\mathrm{Sp}_{2g}(\mathbb{Z})$ in $\mathrm{Sp}_{2g}(\mathbb{C})$. This complex representation is obviously defined over the rational numbers so we may apply [5, 4.8] with field of definition \mathbb{Q} . We thus conclude that $r_{2i} \in H^{2i}(\mathcal{A}_g, \mathbb{Z}_\ell(i))$ is killed $\ell^{\alpha(i)}$, where $\alpha(i)$ is defined as

$$\inf_{\lambda \in H} v_\ell(\lambda^i - 1)$$

and $H \subseteq \mathbb{Z}_\ell^*$ is the image of the Galois group of the field of definition of the cyclotomic character. However, as the base field is \mathbb{Q} this image is all \mathbb{Z}_ℓ^* and the result follows from the definition of n_i . \square

We now aim to show that this upper bound is the precise order of the r_i . We will do this by pulling them back to classifying spaces for certain finite abelian groups in whose cohomology we will be able to determine their images. Over the complex numbers this can be done directly by mapping these finite groups into $\mathrm{Sp}_{2g}(\mathbb{Z})$ and using that the cohomology of \mathcal{A}_g equals the cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$. In the positive characteristic case, if one knew that the specialisation map in the cohomology of \mathcal{A}_g induced an isomorphism then the positive characteristic result would reduce to the characteristic 0 one. Though such a specialisation result seems rather straightforward using the proper and smooth base change theorems, the toroidal compactifications of Chai and Faltings and an induction on g we know of no reference and rather than carrying such an argument through will use another argument. As a motivation for that argument let us begin by noting that any finite subgroup G of $\mathrm{Sp}_{2g}(\mathbb{Z})$ arises as the automorphism group of a principally polarised g -dimensional abelian variety as it has a fixed point on Siegel's upper half

¹ ℓ of course being a prime different from the characteristic

space. Such an abelian variety A together with an action of G may be seen as a principally polarised abelian variety over the classifying stack BG of G and hence corresponds to a map $BG \rightarrow \mathcal{A}_g$. Over the complex numbers, this map induces the given map $G \rightarrow Sp_{2g}(\mathbb{Z})$ on fundamental groups but the map makes sense over an arbitrary base field given a principally polarised abelian g -dimensional variety with a G -action and induces a pull back on cohomology.

Proposition 3.2. *Assume that $i \leq g$. The order of r_{2i} is divisible by $n_i/2$ over \mathbb{C} . In general, each prime ℓ different from the characteristic of the base field, r_{2i} in ℓ -adic cohomology has order divisible by the ℓ -part of $n_i/2$.*

Proof. The ℓ -adic part implies the integral cohomology part so we may pick a prime ℓ different from the characteristic of the base field and look at r_{2i} in ℓ -adic cohomology. What we want to show is that if ℓ is odd and $\ell - 1 \mid 2i$ and k is the largest integer such that $\ell^{k-1} \mid 2i$, then r_{2i} has order at least ℓ^k and similarly for $\ell = 2$. We will do this by defining a map $B\mathbb{Z}/\ell^k \rightarrow \mathcal{A}_g$, such that inverse image of the r_{2i} to $H^{4i}(\mathbb{Z}/\ell^k, \mathbb{Z}_\ell)$ has order ℓ^k . Now, $H^{4i}(\mathbb{Z}/\ell^k, \mathbb{Z}_\ell)$ is isomorphic to \mathbb{Z}/ℓ^k and hence an element in it has order ℓ^k if and only if its reduction modulo ℓ is non-zero. As $H^{4i}(\mathbb{Z}/\ell^k, \mathbb{Z}_\ell)/\ell$ injects into $H^{4i}(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$ it is enough to show that the pullback of r_i is non-zero in $H^{4i}(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$. Assume first that ℓ is odd. Consider now any Galois cover $C \rightarrow \mathbb{P}^1$ with Galois group \mathbb{Z}/ℓ^k which is ramified at 0, 1 and ∞ with ramification group of order ℓ^k , ℓ^k , and ℓ respectively (the existence of such a cover follows directly from Kummer theory). By the Hurwitz formula the genus of such a covering fulfills the relation $2g - 2 = -2\ell^k + 2(\ell^k - 1) + \ell^{k-1}(\ell - 1)$, i.e., $2g = \ell^{k-1}(\ell - 1)$. The action of \mathbb{Z}/ℓ^k on $H^1(C, \mathbb{Q}_\ell)$ has to contain a copy of the irreducible representation given by the action of ℓ^k 'th roots of unity as it must be faithful and as $H^1(C, \mathbb{Q}_\ell)$ has dimension $\ell^{k-1}(\ell - 1)$ it must be equal to it. Furthermore, this action gives a map $B\mathbb{Z}/\ell^k \rightarrow \mathcal{M}_g \rightarrow \mathcal{A}_g$. If now $x \in H^2(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$ is the natural generator (the Bockstein of the generator of $H^1(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$ corresponding to the identity map $\mathbb{Z}/\ell \rightarrow \mathbb{Z}/\ell$) then the total Chern class of it is $\prod_{(i,\ell)=1} (1 + ix)$ which equals $(1 + x^{\ell-1})^{\ell^{k-1}} = 1 + x^{\ell^{k-1}(\ell-1)}$. This gives the non-triviality when $g = \ell^{k-1}(\ell - 1)/2$ and when $g > \ell^k(\ell - 1)/2$ we simply add a principally polarised factor on which \mathbb{Z}/ℓ^k acts trivially.

When $\ell = 2$ we may assume that $k > 2$ as the lower bound to be proven for $k \leq 2$ is implied by the one for $k = 3$. We then make essentially the same construction, a Galois cover with group $\mathbb{Z}/2^k$ of \mathbb{P}^1 ramified at three points with ramification groups of order 2^k , 2^k , and 2 respectively of genus $g = 2^{k-3}$. The rest of the argument is identical to the odd case. \square

Remark 3.3. i) It follows from (1.1) that the r_i are torsion already in the Chow groups. Our result gives a lower bound for this order but we don't know if this bound is sharp.

ii) From the complex point of view our geometric construction can be seen simply as constructing an element of order ℓ^k in $\mathrm{Sp}_{\ell^{k-1}(\ell-1)}(\mathbb{Z})$. This can be done directly, in the odd case one may consider the ring of ℓ^k 'th roots of unity $R = \mathbb{Z}[\zeta]$ with the obvious action of \mathbb{Z}/ℓ^k and the symplectic form $\langle \alpha, \bar{\beta} \rangle := \mathrm{Tr}(\alpha\beta(\zeta - \zeta^{-1})^{-\ell^k + \ell^{k-1} + 1})$. This is obviously a symplectic invariant form and that it is indeed an integer-valued perfect pairing follows from the fact that the different of R is the ideal generated by $(\zeta - \zeta^{-1})^{\ell^k - \ell^{k-1} - 1}$.

iii) When $g = \ell^k(\ell - 1)/2$ we actually get a lower bound for the top Chern class of the de Rham cohomology of the universal curve over \mathcal{M}_g . However, there is no direct analogue of the trick of adding a factor with trivial action so this does not give a lower bound for all $g \geq \ell^k(\ell - 1)/2$.

Theorem 3.4. *We have that $r_{2i+1} = 0$ for all i and that the order of r_{2i} in integral (ℓ -adic) cohomology equals (resp. the ℓ -part of) $n_i/2$ for $i \leq g$.*

Proof. This follows immediately from Props. 3.2 and 3.1. \square

Corollary 3.5. *The order of λ_g is at least $n_g/2$.*

Proof. The top Chern class of H_{dR}^1 is λ_g^2 . \square

Remark 3.6. Our upper and lower bounds for r_{2i} are off by a (multiplicative) factor of 2. Furthermore, when $g = 1$ the lower bound is the correct order.

4. A COMPLETE REPRESENTATIVE CYCLE FOR λ_g ON $\mathcal{A}_g \otimes \mathbb{F}_p$

Consider the closed algebraic subset V_0 of $\mathcal{A}_g \otimes \mathbb{F}_p$ of all abelian varieties with p -rank zero. By Koblitz (see [K]) we know that this is a pure codimension g cycle on $\mathcal{A}_g \otimes \mathbb{F}_p$. It is a complete cycle since abelian varieties of p -rank 0 cannot degenerate. Any complete subvariety of \mathcal{A}_g has codimension at least g in \mathcal{A}_g , see [3] and [11]. Let $\tilde{\mathcal{A}}_g$ be a toroidal compactification. One of us has proved (see [3])

Theorem 4.1. *The cycle class of V_0 in $CH_{\mathbb{Q}}^g(\tilde{\mathcal{A}}_g)$ is given by the formula $[V_0] = (p-1)(p^2-1)\cdots(p^g-1)\lambda_g$.*

This shows that a multiple of λ_g is an *effective* cycle in characteristic $p > 0$. It seems unknown whether a non-zero multiple of the class λ_g can be represented by an effective cycle in characteristic zero.

5. A REPRESENTATIVE CYCLE FOR λ_g ON THE SATAKE COMPACTIFICATION

We let \mathcal{A}_g^* be the minimal (‘Satake’) compactification as defined in [2]. The Satake compactification \mathcal{A}_g^* is a disjoint union

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0.$$

There is a natural morphism $q : \tilde{\mathcal{A}}_g \rightarrow \mathcal{A}_g^*$ for every toroidal compactification $\tilde{\mathcal{A}}_g$ of \mathcal{A}_g . In [3] two representative cycles for the class $\lambda_g \in CH_{\mathbb{Q}}^g(\mathcal{A}_g^*)$ were given. We recall the result.

Lemma 5.1. *The class $q_*(\lambda_g)$ in $CH_{\mathbb{Q}}^g(\mathcal{A}_g^*)$ is represented by a multiple of the fundamental class of the boundary $B_g^* = \mathcal{A}_g^* - \mathcal{A}_g$.*

Proof. The class λ_g vanishes up to torsion on \mathcal{A}_g ; for dimension reasons $q_*\lambda_g$ is then represented by a multiple of the fundamental class of B_g^* . \square

Proposition 5.2. *The cycle class $[B_g^*]$ of the boundary is the same in the Chow group $CH_{\mathbb{Q}}^g(\mathcal{A}_g^*)$ as a multiple of the class of the closure of the image of \mathcal{A}_{g-1}^* in \mathcal{A}_g^* under the map $[X] \mapsto [X \times E]$, with E a generic elliptic curve.*

Proof. Consider (for $g > 2$) the space $\mathcal{A}_{g-1,1}$ of products of a principally polarized abelian variety of dimension $g-1$ and an elliptic curve. It is the image of $\mathcal{A}_{g-1} \times \mathcal{A}_1$ in \mathcal{A}_g under a morphism to \mathcal{A}_g which can be extended to a morphism $\mathcal{A}_{g-1}^* \times \mathcal{A}_1^* \rightarrow \mathcal{A}_g^*$. Since \mathcal{A}_1 is the affine j -line we find a rational equivalence between the cycle class of a generic fibre $\mathcal{A}_{g-1}^* \times \{j\}$ with j a generic point on the j -line and a multiple of the fundamental class of the boundary B_g^* . \square

6. A REPRESENTATIVE CYCLE FOR λ_g ON \mathcal{A}'_g

There are several compactifications of $\mathcal{A}_g \otimes k$. We first choose a suitable smooth toroidal compactification $\tilde{\mathcal{A}}_g$ as constructed in [2]. We have a natural map $q : \tilde{\mathcal{A}}_g \rightarrow \mathcal{A}_g^*$. We call the inverse image under q of $\sqcup_{j \leq t} \mathcal{A}_{g-j}$ the moduli space $\tilde{\mathcal{A}}_g^{(t)}$ of rank $\leq t$ degenerations. This space parametrizes semi-stable abelian varieties whose torus-part has rank $\leq t$. Furthermore, we let $\mathcal{A}'_g = \tilde{\mathcal{A}}_g^{(1)}$ be the moduli space of rank 1-degenerations, i.e. the inverse image of $\mathcal{A}_g \sqcup \mathcal{A}_{g-1} \subset \mathcal{A}_g^*$ under the natural map $q : \tilde{\mathcal{A}}_g \rightarrow \mathcal{A}_g^*$. Unlike the higher rank space $\tilde{\mathcal{A}}_g^{(t)}$ for $t \geq 2$ the space \mathcal{A}'_g does *not* depend on a choice $\tilde{\mathcal{A}}_g$ of compactification of \mathcal{A}_g ; it is a *canonical* partial compactification on \mathcal{A}_g . If we want a full compactification then there is not really a unique one, but we must make choices. See [9].

We start by applying the Grothendieck-Riemann-Roch theorem to the structure sheaf $\mathcal{O}_{\tilde{\mathcal{X}}_g}$ of a compactification $\tilde{\mathcal{X}}_g$ as constructed in [2], of the universal semi-abelian variety and the morphism $\pi : \tilde{\mathcal{X}}_g \rightarrow \tilde{\mathcal{A}}_g$. This gives in the Chow groups with rational coefficients

$$ch(\pi_! \mathcal{O}_{\tilde{\mathcal{X}}_g}) = \pi_*(e^0 \mathrm{Td}^\vee(\Omega^1_{\tilde{\mathcal{X}}_g/\tilde{\mathcal{A}}_g})).$$

The relative cotangent sheaf fits in an exact sequence

$$0 \rightarrow \Omega^1_{\tilde{\mathcal{X}}_g/\tilde{\mathcal{A}}_g} \rightarrow \mathbb{E} \rightarrow \mathcal{F} \rightarrow 0.$$

with \mathbb{E} the Hodge bundle and \mathcal{F} a sheaf with support where π is not smooth. Note that by [2] we have

$$\pi^*(\mathbb{E}) = \Omega^1_{\tilde{\mathcal{X}}_g}(\log)/\pi^*(\Omega^1_{\tilde{\mathcal{A}}_g}(\log)),$$

where \log refers to logarithmic poles along the divisors at infinity. We get

$$ch(\pi_!(\mathcal{O}_{\tilde{\mathcal{X}}_g})) = \pi_*(F) \mathrm{Td}^\vee(\mathbb{E})$$

with $F := \mathrm{Td}^\vee(\mathcal{F})^{-1}$. The derived sheaf $\pi_!(\mathcal{O}_{\tilde{\mathcal{X}}_g})$ equals $\wedge^* \mathbb{E} = \sum_{i=0}^g (-1)^i \wedge^i \mathbb{E}$. By the Borel-Serre formula we have $ch(\wedge^* \mathbb{E}) = \lambda_g \mathrm{Td}(\mathbb{E})^{-1}$. Comparing the terms of degree $\leq g$ yields the result of [3] :

Proposition 6.1. *We have $\pi_*(F) = \lambda_g$.*

From now on we work on the moduli space $\mathcal{A}'_g = \tilde{\mathcal{A}}_g^{(1)}$ of rank ≤ 1 degenerations. Let D^0 be the closed subset corresponding to rank 1 degenerations. The divisor D^0 has a morphism to $\phi : D^0 \rightarrow \mathcal{A}_{g-1}$ which exhibits D^0 as a quotient of the universal abelian variety over \mathcal{A}_{g-1} . The fibre over $x \in \mathcal{A}_{g-1}$ is the dual \hat{X}_{g-1} of the abelian variety X corresponding to x . The ‘universal’ semi-abelian variety G over $\hat{\mathcal{X}}_{g-1}$ is

the \mathbb{G}_m -bundle obtained from the Poincaré bundle $P \rightarrow \mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1}$ by deleting the zero-section. We have the maps

$$G = P - \{(0)\} \rightarrow \mathcal{X}_{g-1} \times_{\mathcal{A}_g} \hat{\mathcal{X}}_{g-1} \rightarrow \mathcal{A}_{g-1}.$$

We shall now work out an expression for $\pi_*(F)$. Let B_g be the cycle on \mathcal{A}'_g which is the locus of *trivial* semi-abelian extensions

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{X} \rightarrow X_{g-1} \rightarrow 0$$

The cycle B_g sits in D^0 as the zero section of $\phi : D^0 \rightarrow \mathcal{A}_{g-1}$.

Theorem 6.2. *The Chow class of B_g and that of $\mathcal{A}_{g-1} \times \{j\}$ in $CH_{\mathbb{Q}}^g(\mathcal{A}'_g)$ are both equal to $(-1)^g \lambda_g / \zeta(1-2g)$ with $\zeta(s)$ denoting the Riemann zeta function.*

The fibre over $x \in D^0$ is a compactification \bar{G} of a \mathbb{G}_m -bundle G over an abelian variety X_{g-1} of dimension $g-1$. The points where π is not smooth are exactly the points of $\bar{G} - G$. So globally the locus where π is not smooth is a cycle Δ obtained from glueing the 0-section and the ∞ -section of the \mathbb{P}^1 -bundle associated to the Poincaré bundle P over $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \hat{\mathcal{X}}_{g-1}$. Note that Δ is of codimension 2 and that an étale cover of Δ is given by $\mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1}$.

The normal bundle to Δ on $\mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1}$ is then $N = P \oplus \tau^*(P^{-1})$ with $\tau(x, \hat{x}) = x + \hat{x}, \hat{x}$. (We identify \tilde{X} with X if needed.) We write $\alpha_1 = c_1(P)$ and $\alpha_2 = c_1(\tau^*(P^{-1}))$. On the space of rank ≤ 1 degenerations (an étale cover of which is $\mathcal{A}_g^{(1)} = \mathcal{A}_g \cup \hat{\mathcal{X}}_{g-1}$) we can write $\Delta = \alpha_1 \alpha_2$. Let $i : \Delta \rightarrow \hat{\mathcal{X}}_g$ be the inclusion.

Then if we write

$$\mathrm{Td}^\vee(L) = \frac{c_1(L)}{(e^{c_1(L)} - 1)} = \sum_{k=0}^{\infty} \frac{b_k}{k!} t^k$$

we have (cf. Mumford [10], p. 303):

$$\pi_*(\mathrm{Td}^\vee(O_\Delta^{-1} - 1)) = \pi_*\left(\sum_{k=1}^{\infty} \frac{(-1)^k b_{2k}}{(2k)!} i_* \left(\frac{\alpha_1^{2k-1} + \alpha_2^{2k-1}}{\alpha_1 + \alpha_2} \right)\right). \quad (2)$$

Observe that $P|_{X_{g-1} \times \hat{x}} = L_{\hat{x}}$ and $\tau^*(P^{-1})|_{X_{g-1} \times \hat{x}} = t_{\hat{x}}^*(L_{\hat{x}}^{-1})$. This implies by the Theorem of the Square that $P \otimes \tau^*(P^{-1})|_{X_{g-1} \times \hat{x}}$ is trivial, i.e. we have

$$c_1(N) = c_1(P \otimes \tau^*(P^{-1})) = \alpha_1 + \alpha_2 = \hat{p}^*(\beta)$$

with β a codimension 1 class on $\hat{\mathcal{X}}_{g-1}$ and \hat{p} the projection of $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \hat{\mathcal{X}}_{g-1}$ on the second factor. In order to determine β we restrict to the other fibres ($x \times \hat{X}_{g-1}$) and find (writing $p = p_X$ and $\hat{p} = p_{\hat{X}}$):

$$P = L_x = t_x^* T \otimes T^{-1}$$

and

$$\begin{aligned} \tau^*(P^{-1}) &= \tau^*(m^* T^{-1} \otimes p^* T \otimes \hat{p}^* T) \\ &= 2^* t_x^* T^{-1} \otimes t_x^* T \otimes T \\ &= L_x^{-1} \otimes (t_x^* T^{-1} \otimes (-1)^* t_x^*(T)^{-1}) \end{aligned}$$

We find (assuming that the Θ -divisor is symmetric)

$$\beta = -2T \quad \text{on } \hat{X}.$$

and

$$N = P \oplus P^{-1} \otimes (\hat{p}^*(O(-2T))). \quad (3)$$

We can consider this as a global identity on Δ . The line bundle T restricts in each fibre \hat{X} to the theta divisor. Developing the terms in (2) we get expressions

$$\begin{aligned} \pi_*(i_*(\alpha_1 + \alpha_2)^r(\alpha_1\alpha_2)^s)) &= \pi_*(i_*(\hat{p}^*(\beta^r)(\alpha_1\alpha_2)^s)) \\ &= j_*(\beta^r \pi'_*(\Delta^s)), \end{aligned}$$

where π' is the restriction to the boundary of π and $j : D \rightarrow \mathcal{A}_g^{(1)}$ is the inclusion of the boundary and where we use $\pi i = \hat{p}$.

For dimension reasons the only surviving terms are of the form $j_*(\beta^r)\pi'_*(\Delta^{g-1})$. Thus the only term that contributes is:

$$\frac{(-1)^g b_{2g}}{(2g)!} \pi_* i_* ((-1)^{g-1} (2g-1) (\alpha_1\alpha_2)^{g-1}).$$

So we need to compute $\pi'_* i_*(\Delta^{g-1}) = \pi_*(\Delta^g)$. The identity $\pi_*(\text{Td}^\vee(\mathcal{F})^{-1}) = \lambda_g$ with

$$F = \text{Td}^\vee(\mathcal{F})^{-1} = 1 + f_2 + f_4 + \dots$$

implies that

$$\pi_*(f_{2g}) = \frac{(-1)b_{2g}}{(2g)!} \pi_* i_* ((2g-1) (\alpha_1\alpha_2)^{g-1}) = \lambda_g.$$

Represent P by the divisor Π on $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \hat{\mathcal{X}}_{g-1}$. Then we have by (3)

$$\alpha_1^{g-1} \alpha_2^{g-1} = (-1)^{g-1} (\Pi^{2g-2} + \sum_{j=1}^{g-2} \binom{g-1}{j} \Pi^{g-1+j} \hat{p}^*(2T)^{g-1-j}).$$

Now apply GRR to the bundle $P \otimes \hat{p}^*(O(nT))$ on $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \hat{\mathcal{X}}_{g-1}$; it says

$$ch(\hat{p}_!(P \otimes \hat{p}^*O(nT))) = \hat{p}_*(e^\Pi) \otimes e^{nT}.$$

But $\hat{p}_!(P \otimes O(nT))$ is a sheaf with support (in codimension $g-1$) over the zero section S . Once again by applying GRR, this time to the inclusion $S \rightarrow \hat{\mathcal{X}}_{g-1}$, we get the relation

$$c_{g-1}(\hat{p}_!(P \otimes O(nT))) = (-1)^{g-2} (g-2)! S$$

It follows that

$$\hat{p}_*(\Pi^{2g-2-j} T^j) = 0 \quad \text{if } j \neq 0$$

and

$$\hat{p}_*(\Pi^{2g-2-j} T^j) = (-1)^{g-1} (2g-2)! [S] \quad \text{if } j = 0.$$

So we find

$$\pi_*(\Delta^g) = j_*(\hat{p}_*(\Pi^{2g-2})) = (-1)^{g-1} (2g-2)! B_g,$$

where B_g is the zero section of $\mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$.

We thus get

$$\begin{aligned}
\lambda_g &= \frac{(-1)b_{2g}}{(2g)!} \pi_* i_* ((2g-1)(\alpha_1 \alpha_2)^{g-1}) \\
&= \frac{(-1)b_{2g}}{(2g)!} (2g-1)(2g-2)!(-1)^{g-1} B_g \\
&= (-1)^g \zeta(1-2g) B_g
\end{aligned}$$

Corollary 6.3. *On the space of rank ≤ 1 degenerations \mathcal{A}'_g we have the formula $\lambda_g = (-1)^g \zeta(1-2g) B_g$ with B_g the locus of semi-stable abelian varieties which are trivial extensions of an abelian variety with \mathbb{G}_m .*

Example 6.4. We have $[B_1] = 12 \lambda_1$ and $[B_2] = 120 \lambda_2$.

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